ISRAEL JOURNAL OF MATHEMATICS, Vol. 51, No. 3, 1985

FINITE RANK OPERATORS WITH LARGE TRACE

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ABSTRACT

We give some estimates for how well the normalized trace functional exposes the identity map on a finite-dimensional space.

Introduction

The identity operator on an n-dimensional space E is an exposed point of the unit ball of $L(E)$. The topic of this paper is estimates for the "exposing modulus" $\delta(t)$; here $\delta(t)$ (Definition 1.1) is the best function with the property that $\|u\|\leqq 1$ and $\operatorname{trace}(u)\geqq n[1-\delta(t)]$ imply $\|1_E - u\|\leqq t$. Our results are divided into three sections; only real spaces are considered.

Section 1 establishes the estimate $t^2/2n^2 \leqq \delta(t) \leqq t$ for any E. An application is that a space with 1-summing constant very close to n must have projection constant close to one.

Section 2 describes the spaces E for which $\delta(t)$ is asymptotically as large as possible, i.e., $\delta(t)\geqq ct$. E has this property iff the group of isometries of E is finite. Another (and equivalent) property these spaces enjoy is that

$$I_1(1_E + w) = n + \operatorname{trace}(w)$$

for any map w on E with integral norm $I_1(w)$ sufficiently small. Finally in section 3 it is shown that, up to constants, $\delta(t)$ behaves like either t or t^2 when E has a 1-unconditional basis.

[†] Research supported by NSF Grant DMS-8320632.
[††] Research supported by NSF Grant DMS-8400960.
Received November 7, 1984

Preliminaries

For E a finite-dimensional real normed space, $L(E)$ denotes the space of linear maps on E under operator norm. In this finite-dimensional setting the *trace* of $u \in L(E)$ is meaningful and is written $\mathrm{tr}(u)$. For $x \in E$ and functional $x' \in E'$, $x' \otimes x$ is the map taking y to $\langle y, x'\rangle x$. The dual of $L(E)$, $n = \dim E < \infty$, can be described using the *integral norm*. For $w: E \to E$ define

$$l_1(w) = \inf \sum_{i \leq m} \|x_i'\| \, \|x_i\|,$$

where the infimum is taken over all representations $w = \Sigma_{i \leq m} x_i' \otimes x_i$. $I(E)$, the space of linear maps on E under the integral norm, is naturally isometric to $L(E)'$ via the pairing $\langle u, w\rangle = \mathrm{tr}(uw)$. We note that $l_1(1_E) = n$ (cf. [4]).

Later some use will be made of the p-summing norms π_p. The basic properties of l_1, π_1 and π_2 may be found in Pietsch's book [9].

Given a basis $(e_i)_{i \leq n}$ for E, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ an n-tuple of signs and π a permutation of $\{1, 2, \ldots, n\}$, let g_ε and g_π be the linear maps defined by $g_\varepsilon(e_i) = \varepsilon_i e_i$ and $g_\pi(e_i) = e_{\pi(i)}$, $1 \leq i \leq n$. The basis is 1-*unconditional* if each such g_ε is an isometry; the basis is 1-*symmetric* if it is 1-unconditional and in addition each g_π is an isometry.

Section 1

Throughout this paper A, B and E denote finite-dimensional *real* Banach spaces, with E having dimension n.

DEFINITION 1.1. Let $e \in A$ have norm one. For each functional $e' \in A'$ satisfying $\|e'\| = \langle e, e'\rangle = 1$, define two functions as follows.

(a) For $0 \leq t \leq 2$,

$$\delta(t, e') = \inf\{\langle e - x, e'\rangle : \|x\| \leq 1 \text{ and } \|e - x\| \geq t\}.$$

(b) For $0 \leq s < \infty$,

$$\phi(s, e') = \sup\{\|e' + x'\| - 1 - \langle e, x'\rangle : \|x'\| \leq s\}.$$

If the identities of e and e' are clear from the context we write $\delta(t)$ and $\phi(t)$; in particular, this simpler notation is used when $e \in L(E)$ is the identity map and $e' = n^{-1} 1_E \in L(E)'$. Recall that e is said to be *exposed by* e' if $\|x\| \leq 1$ and $\langle x, e'\rangle = 1$ must imply $x = e$; and that e is *strongly exposed by* e' if $\delta(t, e') > 0$ for all $t > 0$. For finite-dimensional A these two notions are equivalent. We will say that e is *sharply exposed by* e' if there is a constant $c > 0$ with $\delta(t, e') \geq ct$ for all t.

The first lemma is similar to the duality between the moduli of convexity and smoothness for a Banach space X (Lindenstrauss, [8]).

LEMMA 1.2. *Let $e \in A$ and $e' \in A'$ be given points with $\|e\| = \|e'\| = \langle e, e' \rangle = 1$.*

(1) *For all s and t, $\delta(t, e') + \phi(s, e') \geqq st$.*

(2) *For all s,*

$$\phi(s) = \sup\{st - \delta(t)\colon 0 \leqq t \leqq 2 \text{ and } \delta(t) \leqq 2s\}.$$

PROOF. Given t let x be any point with $\|x\| \leqq 1$ and $\|e - x\| \geqq t$. Choose $x' \in A'$ with $\|x'\| = 1$ and $\|e - x\| = \langle x - e, x' \rangle$. Then for any s

$$\phi(s) + \langle e - x, e' \rangle \geqq \langle x, e' + sx' \rangle - 1 - \langle e, sx' \rangle + \langle e - x, e' \rangle \geqq st.$$

Taking the infimum gives (1) and one inequality in (2). For the other inequality suppose $\|x'\| \leqq s$. Find $x \in A$ with $\|x\| = 1$, $\|e' + x'\| = \langle x, e' + x' \rangle$ and set $t = \|e - x\|$. Then

$$\begin{aligned} \|e' + x'\| - 1 - \langle e, x' \rangle &= \langle x - e, x' \rangle - \langle e - x, e' \rangle \\ &\leqq |x - e\|s - \delta(\|e - x\|) = st - \delta(t) \end{aligned}$$

and further

$$\begin{aligned} \delta(t) &\leqq \langle e - x, x' \rangle = 1 - \langle x, e' + x' \rangle + \langle x, x' \rangle \\ &\leqq \|e'\| - \|e' + x'\| + s \leqq 2s. \end{aligned}$$

□

THEOREM 1.3. *For each $u \in L(E)$ and real r*

$$l_1(1_E - ru)^2 \leqq n^2(1 + r^2\|u\|^2) - 2nr\,\mathrm{tr}(u).$$

PROOF. It is enough to show the inequality for $r = 1$. Let $\|\ \|_2$ be the inner product norm on E determining the ellipsoid of maximum volume contained in the closed unit ball of E. Write H for E under $\|\ \|_2$ and $v\colon H \to E$, $w\colon E \to H$ for the inclusions. By John's Theorem [6] or its dual version (cf. [7]), $\pi_2(w) \leqq n^{1/2}$ and of course $\|v\| \leqq 1$. Write $\phi = wuv$. The 2-summing norm π_2 coincides with the Hilbert–Schmidt norm for maps on H; thus

$$\begin{aligned} \pi_2(1_H - \phi)^2 &= \mathrm{tr}[(1_H - \phi)(1_H - \phi^*)] \\ &= n + \pi_2(\phi)^2 - 2\,\mathrm{tr}(\phi) \\ &\leqq n + \|u\|^2\|v\|^2\pi_2(w)^2 - 2\,\mathrm{tr}(u) \\ &= n(1 + \|u\|^2) - 2\,\mathrm{tr}(u). \end{aligned}$$

But then

$$\begin{aligned} l_1(1_E - u)^2 &= l_1(v(1_E - \phi)w)^2 \\ &\leqq \|v\|^2 \pi_2(1_H - \phi)^2 \pi_2(w)^2 \\ &\leqq n^2(1 + \|u\|^2) - 2n\,\mathrm{tr}(u). \end{aligned}$$ □

The inequality of Theorem 1.3 for $r = 1$ shows that $1_E \in L(E)$ is exposed by $n^{-1}1_E \in L(E)'$, as is shown in Garling [3]. A quantification of this is

COROLLARY 1.4. *Let E be n-dimensional.*
(1) *for* $0 \leqq t \leqq 2$, $t^2/2n^2 \leqq \delta(t) \leqq t$.
(2) *For* $0 \leqq s \leqq 2/n^2$, $0 \leqq \phi(s) \leqq (sn)^2/2$.

PROOF. The lower bound in (1) follows from the theorem with $r = 1$, together with the fact that $\|v\| \leqq l_1(v)$ for any map v. For the upper bound in (1) simply take $u = (1-t)1_E$; $\|u\| \leqq 1$, $\|1_E - u\| = t$ and $\langle 1_E - u, n^{-1}1_E \rangle = t$. Assertion (2) follows from (1) and Lemma 1.2. □

REMARKS. (a) As functions of t both bounds in part (1) can be asymptotically attained. For example, one may check that

$$t^2/2n \leqq \delta(t) \leqq t^2/n \qquad \text{for } E = l_2^n, \quad \text{and } t < 2^{1/2}$$

and

$$t/2n \leqq \delta(t) \leqq t/n \qquad \text{for } E = l_1^n.$$

(b) The constant in the upper estimates for $\delta(t)$ can be improved if E has a 1-unconditional basis. In fact, let X be a space with 1-unconditional basis (b_i) (finite or infinite) and let (b_i') be the sequence of coefficient functionals.

Let $w \in L(X)'$ be any functional norming 1_X. For each $t \in (0,2)$ and k, let $u_k = 1_X - tb_k' \otimes b_k$; $\|u_k\| \leqq 1$ and $\|1_X - u_k\| \geqq t$, so

$$\begin{aligned} m\delta(t, w) &\leqq \sum_{k \leqq m} \langle 1 - u_k, w \rangle \\ &= t\left\langle \sum_{k \leqq m} b_k' \otimes b_k, w \right\rangle \leqq t \end{aligned}$$

whenever $m \leqq \dim X$. This shows that $\delta(t, w) \leqq t/n$ if $n = \dim X < \infty$, and that 1_X is not strongly exposed if $\dim X = \infty$. In the infinite-dimensional case one may check that $w = \Sigma_{k \geqq 1} 2^{-k} b_k' \otimes b_k$ exposes 1_X.

The *projection constant* and 1-*summing constant* of E are denoted by $\gamma_\infty(E)$

and $\pi_1(E)$, respectively. The facts needed here relating to the γ_∞-norm and π_1-norm may be found in Garling–Gordon [4] and Pietsch [9].

COROLLARY 1.5. *Let E be a space of dimension n with $\pi_1(E) > (n^2-1)^{1/2}$. Then*

$$\gamma_\infty(E)\{1-[n^2-\pi_1(E)^2]^{1/2}\} \leqq n^{-1}\pi_1(E).$$

PROOF. By Pietsch duality [9] there is a map $u: E \to E$ with $\gamma_\infty(u)=1$ and $\pi_1(E)=\mathrm{tr}(u)$. Then $\|u\| \leqq \gamma_\infty(u) \leqq 1$. Applying Theorem 1.3 with $r = n^{-1}\,\mathrm{tr}(u)$,

$$\|1_E - ru\|^2 \leqq l_1(1_E - ru)^2 \leqq n^2(1-r^2).$$

Since the last term is less than 1, ru is invertible and

$$\{1-n(1-r^2)^{1/2}\}\|(ru)^{-1}\| \leqq 1.$$

Also

$$\gamma_\infty(E) = \gamma_\infty(uu^{-1}) \leqq \gamma_\infty(u)\|u^{-1}\| = \|u^{-1}\|,$$

and combining the last two displayed inequalities establishes the corollary. □

REMARKS. (a) Deschaseaux [2] and later Garling [3] showed that $\pi_1(E)=n$ implies $\gamma_\infty(E)=1$; the corollary contains this result.

(b) It is not true that $(\dim E_n)^{-1}\pi_1(E_n) \to 1$ implies $\gamma_\infty(E_n) \to 1$. For example, if $E_n = (l_2^2 \oplus l_\infty^{n-2})_\infty$, then

$$n \geqq \pi_1(E_n) \geqq \pi_1(l_\infty^{n-2}) = n-2$$

but

$$\gamma_\infty(E_n) \geqq \gamma_\infty(l_2^2) = (\pi/2)^{1/2}.$$

Section 2

Here our aim is to describe the spaces E for which $1_E \in L(E)$ is sharply exposed.

An element $w \in L(E)'$ with $l_1(w) = \langle 1_E, w\rangle = 1$ is called a *state*, and S will denote the *state space* (= set of all states) in $L(E)'$. More generally for a subspace $A \subseteq L(E)$, define

$$S(A) = \{w|_A : w \in S\} \subset A'.$$

The *numerical range* of $u \in L(E)$ is the set

$$\mathrm{NR}(u) = \{\langle u, w\rangle : w \in S\} \subset \mathbf{R}.$$

The basic facts needed here about numerical ranges of operators may be found in Bonsall–Duncan [1].

THEOREM 2.1. *For an n-dimensional space E the following are equivalent.*

(1) *There is a constant* $c > 0$ *with the property that* $\|u\| \leqq 1$ *and* $n^{-1}\operatorname{tr}(u) > 1 - ct$ *implies* $\|1_E - u\| \leqq t$.

(2) *There is a constant* $c > 0$ *with the property that if* $l_1(w) \leqq c$ *then* $l_1(1_E + w) = n + \operatorname{tr}(w)$.

(3) *The group of isometries of E is finite.*

THEOREM 2.2. *Let A be a subspace of* $L(E)$ *containing* 1_E, *where* $\dim E < \infty$. *Then* $1_E \in A$ *is sharply exposed in A iff A contains no non-zero operator with numerical range* $\{0\}$.

REMARKS. (a) It should be emphasized that all spaces here are real. Theorem 2.1 fails for complex spaces. For example, let (e_i) be the unit vector basis of complex l_p^n, ω a primitive nth root of unity and w be the diagonal map $w(e_k) = \omega^k e_k$; w has trace zero and $l_1(1 + tw) = \Sigma_{k \leqq n} |1 + t\omega^k| > n$ for all $t \neq 0$.

(b) Let B be a real, finite-dimensional normed algebra with identity e, and let $R : B \to L(B)$ be the representation $R_a(b) = ab$. Applying Theorem 2.2 to $A = R(B)$ shows that e is sharply exposed iff B has no non-zero element with numerical range zero.

The proofs of the theorems will be divided into several lemmas. Additionally, frequent use will be made of the following theorem of Lumer and Phillips (cf. [1], page 30).

LUMER–PHILLIPS THEOREM. *Let* $u \in L(X)$. $\operatorname{NR}(u) \subset (-\infty, 0]$ *if and only if* $\exp(tu)$ *is a contraction for all* $t \geqq 0$.

Below A denotes a subspace of $L(E)$ containing the identity map. It is convenient to write e for 1_E and e' for the sate $n^{-1}1_E \mid A$.

LEMMA 2.3. *If F is a face of* $S(A)$ *containing* e', *then* $F = S(A)$.

PROOF. The basic case is $A = L(E)$. To get a contradiction, suppose $F \subset S$ contains e' and that $F \neq S$. By separation arguments (cf. [5]) there is a $v \in L(E)$ and a real a so that (a) $v \mid S \leqq a$, (b) $v \mid F = a$, and (c) $\langle v, w_0 \rangle < a$ for some $w_0 \in S$. Let $u = v - ae$; (a) implies $\operatorname{NR}(u) \subset (-\infty, 0]$ and by the Lumer–Phillips Theorem $\|\exp(tu)\| \leqq 1$, $t \geqq 0$. By (b) $\operatorname{tr}(u) = n\langle u, e' \rangle = 0$, and so

$$\det(\exp(tu)) = \exp(t \operatorname{tr}(u)) = 1, \qquad t \geqq 0.$$

Now a volume argument shows that $\exp(tu)$ maps the closed unit ball of E onto itself, i.e. $\exp(tu)$ is an isometry. Then

$$1 = \|\exp(tu)^{-1}\| = \|\exp(-tu)\|, \qquad t \geqq 0,$$

so again by the Lumer–Phillips Theorem $\mathrm{NR}(-u) \subset (-\infty, 0]$, contradicting (c).

More generally let $F \subset S$ be a face containing $e' \mid A$.

$$M = \{w \in S \colon w \mid A \in F\}$$

is a face of S containing e', and so $M = S$. □

DEFINITION 2.4. Let A be a subspace of $L(E)$ containing e.

(1) $V(A)$ denotes the set of functionals $w \in A'$ having the following property: there is an $a > 0$ with $\|e' + tw\|_{A'} = 1 + t\langle e, w\rangle$ whenever $|t| \leqq a$.

(2) $\mathscr{L}(A) = \{u \in A : \mathrm{NR}(u) = \{0\}\}$.

For $A = L(E)$ we write simply V and $\mathscr{L}$; following Rosenthal [12], $\mathscr{L}$ is called the Lie Algebra of E. Note that both $V(A)$ and $\mathscr{L}(A)$ are vector spaces, with $e' \in V(A)$.

In the lemmas which follow $\| \ \|$ denotes the dual norm on A'.

LEMMA 2.5. *$V(A) = \operatorname{span} S(A)$ and the annihilator of $V(A)$ is $V(A)^0 = \mathscr{L}(A)$.*

PROOF. The main point is that $S(A) \subset V(A)$. To see this let $F = S(A) \cap V(A)$. Since $e' \in F$, we need only show that F is a face of $S(A)$. Let $w_1, w_2 \in S(A)$ and suppose the convex combination $w = \lambda w_1 + (1 - \lambda) w_2 \in F$. Since $V(A)$ is a vector space containing e', $e' - w \in V(A)$. Thus there is an $a > 0$ with

$$\|e' + a(e' - w)\| = 1 + a\langle e, e' - w\rangle = 1.$$

Choose $\nu \in (0, 1)$ to satisfy $\nu - (1 - \nu)(1 - \lambda)a = 0$ and let $b = \nu\lambda(1 + \lambda)^{-1}$. Since

$$e' + b(e' - w_1) = \nu w_2 + (1 - \nu)[e' + a(e' - w)],$$

$$\|e' + t(e' - w_1)\| \leqq 1, \qquad 0 \leqq t \leqq b.$$

The last inequality holds by convexity for $-1 < t \leqq 0$ since both e' and w_1 are in $S(A)$. But also $\|e' + t(e' - w_1)\| \geqq \langle e, e' + t(e' - w_1)\rangle = 1$, so $w_1 \in V(A)$. Similarly, $w_2 \in V(A)$. This shows F is a face, so $F = S(A)$ by Lemma 2.3 and thus $S(A) \subset V(A)$.

For the other containment it is clear that $\mathscr{L}(A)$ is the annihilator of $\operatorname{span} S(A)$. Let $u \in \mathscr{L}(A)$. If $w \in V(A)$ there is a $a > 0$ with

$e' \pm a(e' - w) \in S(A)$. Then $0 = \langle u, e' \rangle \pm a\langle u, e' - w \rangle$ and hence $0 = \langle u, e' \rangle = \langle u, w \rangle$. This shows

$$[\operatorname{span} S(A)]^0 = \mathscr{L}(A) \subset V(A)^0,$$

which completes the proof. □

LEMMA 2.6. *There is a constant a with the following property: for any $w \in V(A)$ and $|t| \leqq a\|w\|$, $\|e' + tw\| = 1 + t\langle e, w \rangle$.*

PROOF. Let $(w_i)_{i \leqq m} \subset V(A)$ be any normalized basis. Since all norms on the finite-dimensional space $V(A)$ are equivalent, there is a constant $c > 0$ with $\Sigma_i |t_i| \leqq c \|\Sigma_i t_i w_i\|$ for every choice of scalars t_i. Choose a_i so that $\|e' + tw_i\| = 1 + t\langle e, w_i \rangle$ whenever $|t| < a_i$. It is easy to check that $a = (\min a_i)/2c$ has the desired property. □

LEMMA 2.7. (1) *If $w \in S(A)$, $a > 1$ and $e' + a(e' - w) \in S(A)$, then*

$$(1 + a^{-1})\delta(t) \geqq \delta(t, w), \qquad 0 \leqq t \leqq 2.$$

(2) *There is a constant $b > 0$ so that, for all $w \in S(A)$,*

$$b\delta(t) \geqq \delta(t, w), \qquad 0 \leqq t \leqq 2.$$

PROOF. For (1), suppose $u \in A$, $\|u\| \leqq 1$ and $\|e - u\| \geqq t$. Then

$$1 \geqq \langle u, e' + a(e' - w) \rangle = 1 - (1 + a)\langle e - u, e' \rangle + a\langle e - u, w \rangle,$$

or

$$(1 + a^{-1})\langle e - u, e' \rangle \geqq \langle e - u, w \rangle.$$

Taking the infimum over all such u gives (1). For (2), let a be given by Lemma 2.6. Since $e', w \in S(A) \subset V(A)$, $\|e' + a(e' - w)\| = 1$; (2) now follows from (1).

PROOF OF 2.1 AND 2.2. First note the following equivalences.

(a) $\delta(t) \geqq ct$ for some c iff

(b) $\phi(s) = 0$ for $0 < s < c_1$ (by Lemma 1.2) iff

(c) $V(A) = A'$ (by Lemma 2.6) iff

(d) $\mathscr{L}(A) = V(A)^0 = (0)$ (by Lemma 2.5).

To prove Theorem 2.2 suppose $e \in A$ is sharply exposed. By Lemma 2.7(2) e must be sharply exposed by e'; this combined with the equivalence of (a) and (b) is the proof.

Since (a), (b) and (d) are equivalent, to verify Theorem 2.1 we need only show that (3) is equivalent to (d). Suppose G, the group of isometries of E, is finite, and

$u \in \mathscr{L}$. By the Lumer–Phillips Theorem, $\exp(tu)$ is an isometry for all t. Thus $\exp(tu) = e$ for small $|t|$, and so $u = 0$. For the converse assume G is infinite. The exponential function maps $\mathscr{L}$ onto G_0, the connected component of $e \in G$ (cf. Robbin [10]). Since G_0 is non-trivial, $\mathscr{L} \neq (0)$. □

REMARK. The Lie algebra $\mathscr{L}$ is closed under the bracket operation $[u, v] = uv - vu$. Together with Lemma 2.5 this implies two permanence properties of V. Notice that $\mathrm{tr}(u[v, w]) = \mathrm{tr}([u, v]w)$. Thus

(a) if v is in the commutator of $\mathscr{L}$, then $[v, w] \in V$ for all $w \in L(E)$, and

(b) if $v \in \mathscr{L}$ then $[v, w] \in V$ for all $w \in V$.

Section 3

Here we consider the modulus $\delta(t)$ when E has special structure, either a 1-unconditional or 1-symmetric basis.

First, Lemma 2.7(2) shows that, up to constants, no state w exposes 1_E better than the normalized trace $n^{-1}1_E$. In the unconditional basis case firmer estimates than Lemma 2.7 hold.

THEOREM 3.1. *Let E be an n-dimensional space and w any state.*

(1) If E has a 1-unconditional basis then $n\delta(t) \geqq \delta(t, w)$ for all t.

(2) If E has a 1-symmetric basis then $\delta(t) \geqq \delta(t, w)$ for all t.

THEOREM 3.2. *Let E be a space with 1-unconditional basis. There is a constant c with either $\delta(t) \geqq ct$ for all t or $\delta(t) \leqq ct^2$ for all t.*

THEOREM 3.3. *Let E be an n-dimensional space with a 1-symmetric basis. Then either $E = l_2^n$ isometrically or 1_E is sharply exposed.*

PROOF OF 3.1. Fix $t \in (0, 2)$, let $(e_i)_{i \leqq n}$ be a 1-unconditional basis, $(e_i')_{i \leqq n}$ be the coefficient functionals and G be the group of isometries of E. First note two simple properties of δ:

(a) $\delta(t, \cdot)$: $S \to \mathbf{R}$ is concave and upper semi-continuous, and

(b) $\delta(t, w) = \delta(t, gwg^{-1})$ for all $w \in S$ and $g \in G$.

Let dg be the normalized Haar measure on G and set $w_0 = \int gwg^{-1}dg$. Using (a) and (b),

$$\delta(t, w) = \int \delta(t, gwg^{-1})dg \leqq \delta(t, w_0).$$

The average w_0 is a state because $\mathrm{tr}(w_0) = \mathrm{tr}(w) = 1$ and $l_1(w_0) \leqq l_1(w) = 1$. Further, $w_0 \in G_c$ = the commutator of G.

Since the basis is 1-unconditional and $w_0 \in G_c$, w_0 must be a diagonal map $w_0 = \sum_{i \leqq n} \lambda_i e'_i \otimes e_i$. For each i, $0 \leqq \lambda_i \leqq 1$, because

$$1 = \operatorname{tr}(w_0) = \sum_i \lambda_i \leqq \sum |\lambda_i| \leqq l_1(w_0) = 1.$$

Let $a = (n-1)^{-1}$. In the notation of Lemma 2.7,

$$e' + a(e' - w_0) = a \sum_{i \leqq n} (1 - \lambda_i) e'_i \otimes e_i$$

has integral norm $a(n-1) = 1$, and so by Lemma 2.7

$$n\delta(t) \geqq \delta(t, w_0) \geqq \delta(t, w).$$

In the case the basis is 1-symmetric $G_c = \mathbf{R}\, 1_E$. Thus $w_0 = \lambda\, 1_E$, $n\lambda = \operatorname{tr}(w_0) = 1$ and so $w_0 = n^{-1} 1_E$. □

Proofs of 3.2 and 3.3. The idea is that if 1_E is not sharply exposed then $\mathcal{L} \neq (0)$; as mentioned above $u \in \mathcal{L}$ means $\exp(tu)$ is an isometry of E for all t. Let $(\ ,\ )$ be any inner product on E invariant under every isometry of E. Note that $u \in \mathcal{L}$ implies $u^* = -u$ with respect to this inner product, and that the basis vectors e_i are orthogonal since the basis is 1-unconditional.

Assume (e_i) is 1-unconditional, let $u \in \mathcal{L}$ be non-zero and suppose for convenience that $a = \langle u(e_1), e'_2 \rangle \neq 0$. Let H be the group of isometries of form g_ε, where $\varepsilon_1 = \varepsilon_2 = 1$. The average $v = |H|^{-1} \sum_{g \in H} gug^{-1}$ is in the Lie algebra $\mathcal{L}$ and $v = a[e'_2 \otimes e_1 - e'_1 \otimes e_2]$ because $v^* = -v$ and v commutes with each $g_\varepsilon \in H$. For each t, $\exp(tv)$ is a $\|\ \|$-isometry whch maps $F = \operatorname{span}(e_1, e_2)$ onto itself, and is also an isometry for the Hilbertian norm $\|x\|_2 = (x, x)^{1/2}$. Since the maps $\exp(tv)$, $t \in \mathbf{R}$, give all rotations on F, $F = l_2^2$ isometrically and each rotation r_θ of F extends to an isometry g_θ of E satisfying $g_\theta \mid \operatorname{span}(e_3, \ldots, e_n) =$ identity. Given t, $0 \leqq t \leqq 2^{1/2}$, choose θ with $2 - 2\cos\theta = t^2$. If g_θ is as described above, $\|g_\theta\| = 1$,

$$\|1_E - g_\theta\| \geqq \|(1_E - g_\theta) \mid F\| = (2 - 2\cos\theta)^{1/2} = t$$

and so

$$\delta(t) \leqq n^{-1} \operatorname{tr}(1_E - g_\theta) = n^{-1}(2 - 2\cos\theta) = t^2/n.$$

This proves Theorem 3.2.

To finish the proof of 3.3, argue as above that if $\mathcal{L} \neq (0)$, then some map of form $e'_s \otimes e_t - e'_t \otimes e_s$, $s \neq t$, must be in $\mathcal{L}$. But then by symmetry of the basis every such map lies in $\mathcal{L}$, so $\mathcal{L} = \{u : u^* = -u\}$. It's well known that every

(,)-orthogonal map g has form $g = \varepsilon \exp(u)$, where $u^* = -u$ and $|\varepsilon| = 1$. Then every orthogonal map is a $\|\ \|$-isometry of E and, since the orthogonal maps act transitively on the sphere $\{x : \|x\|_2 = 1\}$, $E = l_2^n$ isometrically. □

REMARKS. (a) For any space E the space V has dimension at least $n(n+1)/2$. As above, each element of $\mathscr{L}$ is skew-symmetric and by Lemma 2.5 each symmetric map is in V.

(b) Theorem 3.3 can also be proven by combining Theorem 2.1 with the description of the group of isometries of a 1-symmetric space given in [11].

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